

# ORTHOGONAL APARTMENTS IN HILBERT GRASSMANNIANS

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ABSTRACT. Let  $H$  be an infinite-dimensional complex Hilbert space and let  $\mathcal{L}(H)$  be the logic formed by all closed subspaces of  $H$ . For every natural  $k$  we denote by  $\mathcal{G}_k(H)$  the Grassmannian consisting of  $k$ -dimensional subspaces. An orthogonal apartment of  $\mathcal{G}_k(H)$  is the set consisting of all  $k$ -dimensional subspaces spanned by subsets of a certain orthogonal base of  $H$ . Orthogonal apartments can be characterized as maximal sets of mutually compatible elements of  $\mathcal{G}_k(H)$ . We show that every bijective transformation  $f$  of  $\mathcal{G}_k(H)$  such that  $f$  and  $f^{-1}$  send orthogonal apartments to orthogonal apartments (in other words,  $f$  preserves the compatibility relation in both directions) can be uniquely extended to an automorphism of  $\mathcal{L}(H)$ .

## 1. INTRODUCTION

The concept of apartment comes from the theory of Tits buildings [7], combinatorial constructions related to groups of various types. A *building* can be defined as an abstract simplicial complex together with a collection of subcomplexes called *apartments* and satisfying some axioms. One of these axioms says that all apartments are isomorphic to the simplicial complex associated to a certain Coxeter system. This Coxeter system defines the building type. The vertex set of a building is the disjoint union of so-called *building Grassmannians* [3, 5]. The intersections of apartments with a building Grassmannian are called apartments of this Grassmannian. For every building of classical type all bijective transformations of Grassmannians sending apartments to apartments can be uniquely extended to building automorphisms [3].

Every building of type  $A_{n-1}$  is the flag complex of a certain  $n$ -dimensional vector space  $V$ . The associated Grassmannians are  $\mathcal{G}_k(V)$ ,  $k \in \{1, \dots, n-1\}$  formed by  $k$ -dimensional subspaces of  $V$ . Every apartment of  $\mathcal{G}_k(V)$  is related to a certain base of  $V$  and consists of all  $k$ -dimensional subspaces spanned by subsets of this base. For this case the above mentioned statement on apartments preserving transformations can be formulated as follows: every bijective transformation of  $\mathcal{G}_k(V)$  sending apartments to apartments is induced by a semilinear automorphism of  $V$  or a semilinear isomorphism of  $V$  to the dual vector space  $V^*$  and the second possibility can be realized only for  $n = 2k$ . A more general result related to isometric embeddings of Grassmann graphs can be found in [4, Chapter 5]. Apartments preserving transformations of Grassmannians formed by subspaces with infinite both dimension and codimension are described in [3, Theorem 3.18].

It is natural to ask what is happening if we take Grassmannians of Hilbert spaces together with *orthogonal apartments*, i.e. apartments defined by orthogonal bases?

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Let  $H$  be an infinite-dimensional complex Hilbert space. Denote by  $\mathcal{L}(H)$  the logic formed by all closed subspaces of  $H$ . This logic plays an important role in mathematical foundations of quantum theory if  $H$  is separable (see, for example, [8]), but we will consider an arbitrary infinite-dimensional Hilbert space. It is well-known that every automorphism of  $\mathcal{L}(H)$  is induced by a unitary or conjugate-unitary operator on  $H$ .

Recall that two closed subspaces  $X, Y \subset H$  are *compatible* if there exist closed subspaces  $X', Y'$  such that  $X \cap Y, X', Y'$  are mutually orthogonal and

$$X = X' + (X \cap Y), \quad Y = Y' + (X \cap Y).$$

Theorem 2.8 from [2] can be reformulated as follows: if  $f$  is a bijective transformation of  $\mathcal{L}(H)$  preserving the compatibility relation in both directions then there exists a unitary or conjugate-unitary operator  $U$  such that for every  $X \in \mathcal{L}(H)$  we have  $f(X) = U(X)$  or  $f(X)$  coincides with the orthogonal complement of  $U(X)$  (the second possibility is related with the fact that if  $X$  and  $Y$  compatible then the orthogonal complement of  $X$  is compatible to  $Y$ ). Note that the compatibility is a property distinguishing classical logics from quantum: in a classical logic any two elements are compatible, a quantum logic contains non-compatible elements.

For every natural  $k$  we denote by  $\mathcal{G}_k(H)$  the Grassmannian formed by  $k$ -dimensional subspaces of  $H$ . In contrast to Grassmannians of vector spaces, where for any two elements there is an apartment containing them, two elements of  $\mathcal{G}_k(H)$  are contained in the same orthogonal apartment if and only if they are compatible. Moreover, orthogonal apartments can be characterized as maximal sets of mutually compatible elements. Therefore, a bijective transformation of  $\mathcal{G}_k(H)$  preserves the class of orthogonal apartments in both directions if and only if it is preserving the compatibility relation in both directions. We show that every such transformation can be uniquely extended to an automorphism of the logic  $\mathcal{L}(H)$ .

The proof is purely combinatorial and based on a modification of the idea given in [4, Section 5.2]. Also, we use the following fact: every bijective transformation of  $\mathcal{G}_k(H)$  preserving the orthogonality relation in both directions can be uniquely extended to an automorphism of  $\mathcal{L}(H)$  [1, 6]. For  $k = 1$  this statement is known as Wigner's theorem (see, for example, [8, Theorem 4.29]).

## 2. RESULT

The *orthogonal apartment* of  $\mathcal{G}_k(H)$  associated to an orthogonal base of  $H$  is the set consisting of all  $k$ -dimensional subspaces spanned by subsets of this base. Every orthogonal apartment can be obtained from the unique orthonormal base and the other related bases are formed by scalar multiples of the vectors from this base.

**Proposition 1.** *The class of orthogonal apartments coincides with the class of maximal sets of mutually compatible elements of  $\mathcal{G}_k(H)$ .*

*Proof.* It is easy to see that every orthogonal apartment is a maximal set of mutually compatible elements. Let  $\mathcal{X}$  be a subset of  $\mathcal{G}_k(H)$ , where any two elements are compatible. Denote by  $\mathcal{X}'$  the set consisting of all minimal non-zero intersections of elements from  $\mathcal{X}$ . Any two distinct elements of  $\mathcal{X}'$  are orthogonal. It is clear that  $\mathcal{X}$  is a maximal set of mutually compatible elements if and only if all elements of  $\mathcal{X}'$  are 1-dimensional subspaces and non-zero vectors lying on them form an orthogonal base of  $H$ .  $\square$

**Remark 1.** Similarly, every maximal set of mutually compatible elements of  $\mathcal{L}(H)$  is formed by all closed subspaces spanned by subsets of a certain orthogonal base.

By Proposition 1, a bijective transformation of  $\mathcal{G}_k(H)$  preserves the class of orthogonal apartments in both directions if and only if it is preserving the compatibility relation in both directions.

**Theorem 1.** *Let  $f$  be a bijective transformation of  $\mathcal{G}_k(H)$  such that  $f$  and  $f^{-1}$  send orthogonal apartments to orthogonal apartments, in other words,  $f$  preserves the compatibility relation in both directions. Then  $f$  can be uniquely extended to an automorphism of  $\mathcal{L}(H)$ .*

Two elements of  $\mathcal{G}_1(H)$  are compatible if and only if they are orthogonal, i.e. for  $k = 1$  our result coincides with the mentioned above Wigner's theorem.

### 3. PROOF

We prove Theorem 1 for  $k \geq 2$ . Let  $\mathcal{A}$  be the orthogonal apartment of  $\mathcal{G}_k(H)$  defined by an orthogonal base  $\{e_i\}_{i \in I}$ . For every  $i \in I$  we denote by  $\mathcal{A}(+i)$  and  $\mathcal{A}(-i)$  the sets consisting of all elements of  $\mathcal{A}$  which contain  $e_i$  and do not contain  $e_i$ , respectively. Also, we set

$$\mathcal{A}(+i, +j) := \mathcal{A}(+i) \cap \mathcal{A}(+j),$$

$$\mathcal{A}(+i, -j) := \mathcal{A}(+i) \cap \mathcal{A}(-j),$$

$$\mathcal{A}(-i, -j) := \mathcal{A}(-i) \cap \mathcal{A}(-j)$$

for any distinct  $i, j \in I$ .

We say that a subset of  $\mathcal{A}$  is *inexact* if there is an orthogonal apartment distinct from  $\mathcal{A}$  and containing this subset.

**Example 1.** For any distinct  $i, j \in I$  the subset

$$(1) \quad \mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j)$$

is inexact. In the base  $\{e_i\}_{i \in I}$  we replace the vectors  $e_i, e_j$  by any other pair of orthogonal vectors belonging to the 2-dimensional subspace  $\mathbb{C}e_i + \mathbb{C}e_j$  and consider the associated orthogonal apartment  $\mathcal{A}'$ . Then

$$\mathcal{A} \cap \mathcal{A}' = \mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j).$$

**Lemma 1.** *Every inexact subset of  $\mathcal{A}$  is contained in a subset of type (1), i.e. every maximal inexact subset is of type (1).*

*Proof.* Let  $\mathcal{X}$  be an inexact subset of  $\mathcal{A}$ . For every  $i \in I$  we denote by  $S_i$  the intersection of all subspaces  $X$  which satisfy one of the following conditions:

- $X$  is an element of  $\mathcal{X}$  containing  $e_i$ ,
- $X$  is the orthogonal complement of an element from  $\mathcal{X}$  non-containing  $e_i$ .

Each  $S_i$  is non-zero. If every  $S_i$  is 1-dimensional then  $\mathcal{A}$  is the unique orthogonal apartment containing  $\mathcal{X}$  which contradicts the fact that  $\mathcal{X}$  is inexact. So, there is at least one  $i \in I$  such that  $\dim S_i \geq 2$ . We take any  $j \neq i$  such that  $e_j$  belongs to  $S_i$ . Then

$$\mathcal{X} \subset \mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j).$$

Indeed, if  $X \in \mathcal{X}$  contains  $e_i$  then  $e_j \in S_i \subset X$  and  $X$  belongs to  $\mathcal{A}(+i, +j)$ . If  $X \in \mathcal{X}$  does not contain  $e_i$  then  $e_j \in S_i \subset X^\perp$  which means that  $e_j$  is not contained in  $X$  and  $X$  belongs to  $\mathcal{A}(-i, -j)$ .  $\square$

A subset  $\mathcal{C} \subset \mathcal{A}$  is said to be *complementary* if  $\mathcal{A} \setminus \mathcal{C}$  is a maximal inexact subset, i.e.

$$\mathcal{A} \setminus \mathcal{C} = \mathcal{A}(+i, +j) \cup \mathcal{A}(-i, -j)$$

for some distinct  $i, j \in I$ . Then

$$\mathcal{C} = \mathcal{A}(+i, -j) \cup \mathcal{A}(+j, -i).$$

This complementary subset will be denoted by  $\mathcal{C}_{ij}$ . Note that  $\mathcal{C}_{ij} = \mathcal{C}_{ji}$ .

**Lemma 2.** *Two  $k$ -dimensional subspaces  $X, Y \in \mathcal{A}$  are orthogonal if and only if the number of complementary subsets of  $\mathcal{A}$  containing both  $X$  and  $Y$  is finite.*

*Proof.* If the complementary subset  $\mathcal{C}_{ij}$  contains both  $X$  and  $Y$  then one of the following possibilities is realized:

- (1) one of  $e_i, e_j$  belongs to  $X \setminus Y$  and the other to  $Y \setminus X$ ,
- (2) one of  $e_i, e_j$  belongs to  $X \cap Y$  and the other is not contained in  $X + Y$ .

The number of complementary subsets  $\mathcal{C}_{ij}$  satisfying (1) is finite. If  $X$  and  $Y$  are orthogonal then  $X \cap Y = 0$  and there is no  $\mathcal{C}_{ij}$  satisfying (2). In the case when  $X \cap Y \neq 0$ , there are infinitely many such  $\mathcal{C}_{ij}$ .  $\square$

Let  $f$  be a bijective transformation of  $\mathcal{G}_k(H)$  such that  $f$  and  $f^{-1}$  send orthogonal apartments to orthogonal apartments. For any orthogonal  $k$ -dimensional subspaces  $X, Y \subset H$  there is an orthogonal apartment  $\mathcal{A} \subset \mathcal{G}_k(H)$  containing them. It is clear that  $f$  sends inexact subsets of  $\mathcal{A}$  to inexact subsets of  $f(\mathcal{A})$ . Similarly,  $f^{-1}$  transfers inexact subsets of  $f(\mathcal{A})$  to inexact subsets of  $\mathcal{A}$ . This implies that  $\mathcal{X}$  is a maximal inexact subset of  $\mathcal{A}$  if and only if  $f(\mathcal{X})$  is a maximal inexact subset of  $f(\mathcal{A})$ . Therefore, a subset of  $\mathcal{A}$  is complementary if and only if its image is a complementary subset of  $f(\mathcal{A})$ . Then Lemma 2 guarantees that  $f(X)$  and  $f(Y)$  are orthogonal. Similarly, we establish that  $f^{-1}$  transfers orthogonal subspaces to orthogonal subspaces. So,  $f$  preserves the orthogonality relation in both directions which means that it can be uniquely extended to an automorphism of  $\mathcal{L}(H)$ .

#### 4. FINAL REMARKS

Consider the case when  $H$  is a complex Hilbert space of finite dimension  $n$ . As above, we write  $\mathcal{G}_k(H)$  for the Grassmannian of  $k$ -dimensional subspaces of  $H$ . Let  $\mathcal{A}$  be an orthogonal apartment of  $\mathcal{G}_k(H)$ . In almost all cases, the dimension of the intersection of two elements from  $\mathcal{A}$  can be characterized in terms of complementary subsets and there is an analogue of Theorem 1 (we get a statement similar to [2, Theorem 2.8] if  $n = 2k$ ). For  $n = 2k \pm 2$  such a characterization is impossible and we need some additional arguments.

For example, if  $n = 6$  and  $k = 2$  then  $|\mathcal{A}| = 15$  and the intersection of two distinct elements from  $\mathcal{G}_k(H)$  is zero or 1-dimensional. Any pair of elements from  $\mathcal{A}$  is contained in precisely 4 distinct complementary subsets. The intersection of two complementary subsets always is a 3-element subset and we cannot distinguish one pair of complementary subsets from others.

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